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Branching rules and even-dimensional rotation groups SO_{2k}

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Abstract. Unambiguous methods are developed for calculating branching rules for the classical subgroups of the even-dimensional rotation group SO_{2k}. Complete results are given for the subgroups $SU_k \times U_1$, $SO_{2k-2} \times U_1$, $SO_{2p} \times SO_{2q}$ and $SO_{2p+1} \times SO_{2q+1}$. A number of examples relevant to problems in supergravity and unification theories are given. A complete resolution of the antisymmetric powers of the basic spinor irrep of SO_{10} is given and the results extended to SO_{11}.

1. Introduction

The even-dimensional rotation groups SO_{2k} are finding extensive application in many areas of physics. These applications require a detailed knowledge of the properties of the irreducible representations (irreps) of SO_{2k} and have often been fraught with ambiguities that are associated with the peculiar properties of the irreps of SO_{2k}. Many results have been deduced by trial and error, with some remaining in error. It is the purpose of this paper to present, in an unambiguous way, branching rules for the tensor and spinor irreps of SO_{2k} to important classical subgroups. Their application is illustrated by a number of examples that also serve to correct a number of erroneous or imprecise results already in the literature.

The character theory of the full orthogonal group O_n has been well studied for both spinor and tensor representations (Brauer and Weyl 1935, Murnaghan 1938, Littlewood 1950). The corresponding theory for SO_{2k+1} requires only trivial modifications. The theory for SO_{2k} is complicated by the existence of irreps that, while being non-equivalent, are conjugate to one another under an involutary outer automorphism involving a matrix of determinant -1. These problems may be resolved by use of the properties of the Weyl weight spaces of the irreps but at the expense of obscuring the n dependence of the results. In many cases it is desirable to produce results that hold for all n. Furthermore, there is some advantage to be gained from the construction of explicit formulae for computing branching rules.

Throughout, we shall employ spinor and tensor methods rather than explicit weight-space constructions. These methods have considerable merit in physical applications and lead to a notation that is closer to the tools customarily used by physicists.

Spinor and tensor methods have been used very successfully (King 1975) to derive branching rule formulae for the embedding of one classical Lie group in another. The formulae all involve operations on S functions (Littlewood 1950, Wybourne 1970). King has given extensive results for the groups O_n and SO_{2k+1} but omits any treatment of SO_{2k} noting that these cases can best be dealt with by the method of difference characters (Murnaghan 1938) or weight-space techniques. In view of the unprecedented interest in the groups SO_{2k}, it seems appropriate to extend King's study to the classical subgroups of SO_{2k}.

Our emphasis here is to present new results which are illustrated by examples relevant to problems in grand unification and supergravity theories. Branching rules for the important group–subgroup combinations $SO_{2k} \supset SU_k \times U_1$, $SO_{2k} \supset SO_{2k-2} \times U_1$, $SO_{2(p+q)} \supset SO_{2p} \times SO_{2q}$, and $SO_{2(p+q+1)} \supset SO_{2p+1} \times SO_{2q+1}$ are given for both spinor and tensor irreps.

Branching rules arise in the group–subgroup restriction $G \downarrow H$. The inverse restriction $H \uparrow_r G$ (King 1975) is considered and used to give general results for such cases as $SO_{2k} \uparrow_r SO_{2k+2}$ that arise in supergravity (Curtright 1982a, b). The novel inverse restriction $SO_{2k} \uparrow_r SO_{2k+2} \times SU_2$ is used to count the multiplicities of the SO_{2k+2} irreps in $SO_{2k} \uparrow_r SO_{2k+2}$ using the results of an earlier paper on the replication of irreps (Wybourne 1983).

Finally we discuss the resolution of the Kronecker powers of the basic spinor irrep of SO_{2k} and give a complete procedure for unequivocally resolving all the powers of the **16** irrep of SO_{10}. Furthermore, we are led to a procedure for determining the multiplet content of the $d = 10$ scalar superfield that ensures the equality of dimensions and Dynkin indices for the boson and fermion sectors thus eliminating earlier trial and error methods (Curtright 1982a, b) and correcting earlier results (Bergshoeff and DeRoo 1982).

The symmetrised powers of rotation groups have been reviewed extensively by King *et al* (1981). Complete, and unambiguous, algorithms for resolving the Kronecker products of the rotation groups have also been given (Black *et al* 1983). These two papers (abbreviated henceforth to KLW and BKW respectively) establish much of the notation used in this paper and will be referred to repeatedly. The principal results appear as tables whose derivations rest on the work contained in KLW, BKW and the pioneering work of King (1975).

2. Labelling of irreps

In this paper we shall primarily be interested in the classical compact semi-simple Lie algebras A_k, B_k, C_k, and D_k and their respective Lie groups SU_{k+1}, SO_{2k+1}, Sp_{2k} and SO_{2k}. The labelling of the irreps of these groups has been discussed in KLW and BKW and only essential details are given here. The irreps may be unambiguously labelled by specifying their maximal weights. Equivalent labels may be given either in terms of a set of k non-negative integers $\boldsymbol{a} = (a_1, a_2, \ldots, a_k)$ labelling the nodes of the appropriate Dynkin diagram or in terms of suitably defined partitions $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ (cf Wybourne 1974).

The standard partition labels for the irreps of the classical groups are given in table 1 following the notation given by BKW. The relationship between the partition labels $\boldsymbol{\lambda}$ and the Dynkin labels $\boldsymbol{a}$ is given in table 2 after the manner of King and Al-Qubanchi (1981). In the latter table the irreps of SO_n are assumed to be labelled by partitions with the λ_i either all integers (tensor irreps) or all half-integers (spinor irreps).

We write for SO_{2k} for $\lambda_k > 0$

$$[\lambda]_{\pm} = [\lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \pm\lambda_k]. \qquad (1)$$

Table 1. Standard labels for the irreps of the classical groups of rank k.

Group G	Label	Constraint
U_n	$\{\bar{\mu};\lambda\}$	$p+q\leqslant n=k$
SU_n	$\{\lambda\}$	$p\leqslant n-1=k$
SO_{2k+1}	$[\lambda]$	$p\leqslant k$
	$[\Delta;\lambda]$	$p\leqslant k$
Sp_{2k}	$\langle\lambda\rangle$	$p\leqslant k$
SO_{2k}	$[\lambda]$	$p<k$
	$[\lambda]_+,[\lambda]_-$	$p=k$
	$[\Delta;\lambda]_+,[\Delta;\lambda]_-$	$p\leqslant k$

$\lambda=(\lambda_1\lambda_2\ldots\lambda_p)$ with $\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_p>0$.
$\mu=(\mu_1\mu_2\ldots\mu_q)$ with $\mu_1\geqslant\mu_2\geqslant\ldots\geqslant\mu_q>0$.
λ_i and μ_j are positive integers for $i=1,2,\ldots,p$ and $j=1,2,\ldots q$ respectively.

Table 2. Relationship of Dynkin labels $\boldsymbol{a}$ to partition labels $\boldsymbol{\lambda}$ for the classical Lie groups.

Lie group		
SU_{k+1}	$a_1=\lambda_1-\lambda_2$	$\lambda_1=a_1+a_2+\ldots+a_{k-1}+a_k$
	$a_2=\lambda_2-\lambda_3$	$\lambda_2=a_2+\ldots+a_{k-1}+a_k$
	$\vdots$	$\vdots$
	$a_{k-1}=\lambda_{k-1}-\lambda_k$	$\lambda_{k-1}=a_{k-1}+a_k$
	$a_k=\lambda_k$	$\lambda_k=a_k$
SO_{2k+1}	$a_1=\lambda_1-\lambda_2$	$\lambda_1=a_1+a_2+\ldots+a_{k-1}+\frac{1}{2}a_k$
	$a_2=\lambda_2-\lambda_3$	$\lambda_2=a_2+\ldots+a_{k-1}+\frac{1}{2}a_k$
	$\vdots$	$\vdots$
	$a_{k-1}=\lambda_{k-1}-\lambda_k$	$\lambda_{k-1}=a_{k-1}+\frac{1}{2}a_k$
	$a_k=2\lambda_k$	$\lambda_k=\frac{1}{2}a_k$
Sp_{2k}	$a_1=\lambda_1-\lambda_2$	$\lambda_1=a_1+a_2+\ldots+a_{k-1}+a_k$
	$a_2=\lambda_2-\lambda_3$	$\lambda_2=a_2+\ldots+a_{k-1}+a_k$
	$\vdots$	$\vdots$
	$a_{k-1}=\lambda_{k-1}-\lambda_k$	$\lambda_{k-1}=a_{k-1}+a_k$
	$a_k=\lambda_k$	$\lambda_k=a_k$
SO_{2k}	$a_1=\lambda_1-\lambda_2$	$\lambda_1=a_1+a_2+\ldots+a_{k-2}+\frac{1}{2}a_{k-1}+\frac{1}{2}a_k$
	$a_2=\lambda_2-\lambda_3$	$\lambda_2=a_2+\ldots+a_{k-2}+\frac{1}{2}a_{k-1}+\frac{1}{2}a_k$
	$\vdots$	$\vdots$
	$a_{k-2}=\lambda_{k-2}-\lambda_{k-1}$	$\lambda_{k-2}=a_{k-2}+\frac{1}{2}a_{k-1}+\frac{1}{2}a_k$
	$a_{k-1}=\lambda_{k-1}-\lambda_k$	$\lambda_{k-1}=\frac{1}{2}a_{k-1}+\frac{1}{2}a_k$
	$a_k=\lambda_{k-1}+\lambda_k$	$\lambda_k=\frac{1}{2}a_{k-1}-\frac{1}{2}a_k$

The spinor irreps will often be written for SO_{2k+1} as

$$[\Delta;\lambda]=[\lambda_1+\tfrac{1}{2},\lambda_2+\tfrac{1}{2},\ldots,\lambda_k+\tfrac{1}{2}] \tag{2}$$

and for SO_{2k} as

$$[\Delta;\lambda]_\pm=[\lambda_1+\tfrac{1}{2},\lambda_2+\tfrac{1}{2},\ldots,\pm\lambda_k\pm\tfrac{1}{2}]. \tag{3}$$

Likewise tensor irreps of SO_{2k} having k non-zero parts will often be written as

$$[\Box; \lambda]_{\pm} = [\lambda_1 + 1, \lambda_2 + 1, \ldots, \pm\lambda_k \pm 1] \tag{4}$$

with

$$[\Box; 0]_{\pm} = \Box_{\pm} = [1^{k-1}, \pm 1] = [1^k]_{\pm}. \tag{5}$$

The irreps of the unitary group U_n may be labelled as $\{\lambda\}$ (Littlewood 1950) where the partition λ serves to specify the symmetry properties of the corresponding lth-rank (l being the sum of the parts of λ) covariant tensor forming the basis of this representation. Along with these irreps, there are irreps associated with contravariant tensors labelled $\{\bar{\mu}\}$ and also irreps whose bases are mixed tensors labelled by $\{\bar{\mu}; \lambda\}$. In this notation the λ partition is associated with l covariant indices of the basis tensor while the barred μ partition is associated with m contravariant indices. It is convenient to write

$$\{\bar{0}; \lambda\} = \{\lambda\} \qquad \text{and} \qquad \{\bar{\mu}; 0\} = \{\bar{\mu}\}. \tag{6}$$

The U_n irreps $\{\bar{\mu}; \lambda\}$ may be represented by composite Young diagrams (BKW § 2).

The group U_n possesses a one-dimensional irrep $\varepsilon = \{1^n\}$ that maps each group element onto its determinant with

$$\bar{\varepsilon} = \varepsilon^{-1} = \{\overline{1^n}\}. \tag{7}$$

The product of ε with any other irrep of U_n is also an irrep of U_n and inequivalent irreps related by some power ε^r are said to be associated. If r is half an odd integer then the irrep of U_n is double valued, analogous to the spinor irreps of O_n.

Since under $U_n \downarrow SU_n$ we have $\varepsilon \downarrow \{0\}$ all mutually associated irreps of U_n give equivalent irreps of SU_n under this restriction. Moreover, each inequivalent irrep of SU_n may be denoted by a partition into less than n parts.

3. Modification rules

While the standard partition labels given in table 1 suffice to completely label the irreps of the classical compact Lie groups in many calculations, non-standard labels will arise. The equivalence relations between non-standard and standard labels are known as modification rules (Murnaghan 1938). These have been extensively discussed in BKW.

Modification rules involving simple equivalences are given in table 3(a). Those for SO_{2k} reflect the reducibility of representations labelled by k-part partitions referred to earlier. The double primed symbols correspond to difference characteıs (Murnaghan 1938, BKW § 5, KLW § 4).

The remaining modification rules relevant to this paper reduce the number of parts p in a partition where $p > k$ to yield a standard label or a null result. In each case the modification rules amount to removing from the appropriate Young diagram of the partition a continuous boundary of hook length h starting at the foot of the first column. The relevant set of modification rules of this type are given in table 3(b). These rules may be used repeatedly to yield finally either a signed standard irrep label or a null result. Detailed examples of their application are given in KLW and BKW.

Table 3. Modification rules.

(*a*) Group	Rule	Constraint
SU_n	$\{\lambda\}=\{\bar{\lambda}\}=0$	$p>n$
	$\{\lambda\}=\{\lambda/n^{\lambda_n}\}$	$p=n$
	$\{\bar{\mu};\lambda\}=\{\mu_1+\lambda_1,\mu_1+\lambda_2,\ldots,\mu_1-\mu_3,\mu_1-\mu_2,0\}$	$p+q\leqslant n$
	$\{\bar{\lambda}\}=\{\lambda_1-\lambda_n,\lambda_1-\lambda_{n-1},\ldots,0\}$	$p\leqslant n$
SO_{2k}	$[\lambda]=[\Box;\lambda/1^k]$	$p=k$
	$[\lambda]_\pm=[\Box;\lambda/1^k]_\pm$	$p=k$
	$[\lambda]=[\lambda]_+ + [\lambda]_-$	$p=k$
	$[\lambda]''=[\lambda]_+ - [\lambda]_-$	$p=k$
	$[\Delta;\lambda]=[\Delta;\lambda]_+ + [\Delta;\lambda]_-$	$p\leqslant k$
	$[\Delta;\lambda]''=[\Delta;\lambda]_+ - [\Delta;\lambda]_-$	$p\leqslant k$
	$[\Box;\lambda]=[\Box;\lambda]_+ + [\Box;\lambda]_-$	$p\leqslant k$
	$[\Box;\lambda]''=[\Box;\lambda]_+ - [\Box;\lambda]_-$	$p\leqslant k$

(*b*) Group	Rule	Hook length
U_n, SU_n	$\{\bar{\mu};\lambda\}=(-1)^{c+d-1}\{\overline{\mu-h};\lambda-h\}$	$h=p+q-n-1\geqslant 0$
SO_{2k+1}	$[\lambda]=(-1)^{c-1}[\lambda-h]$	$j=2p-2k-1>0$
	$[\Delta;\lambda]=(-1)^c[\Delta;\lambda-h]$	$h=2p-2k-2\geqslant 0$
Sp_{2k}	$\langle\lambda\rangle=(-1)^c\langle\lambda-h\rangle$	$h=2p-2k-2\geqslant 0$
SO_{2k}	$[\lambda]=(-1)^{c-1}[\lambda-h]$	$h=2p-2k>0$
	$[\Delta;\lambda]=(-1)^c[\Delta;\lambda-h]$	$h=2p-2k-1\geqslant 0$
	$[\Delta;\lambda]''=(-1)^{c-1}[\Delta;\lambda-h]''$	$h=2p-2k-1>0$
	$[\Delta;\lambda]_\pm=(-1)^c[\Delta;\lambda-h]_\mp$	$h=2p-2k-1\geqslant 0$
	$[\Box;\lambda]=(-1)^{c-1}[\Box;\lambda-h]$	$h=2p-2k-2\geqslant 0$
	$[\Box;\lambda]''=(-1)^c[\Box;\lambda-h]''$	$h=2p-2k-2\geqslant 0$
	$[\Box;\lambda]_\pm=(-1)^{c-1}[\Box;\lambda-h]_\mp$	$h=2p-2k-2\geqslant 0$

(λ) and (μ) are partitions of p and q respectively. c and d are columns of (λ) and (μ) in which the boundary hook ends.

4. Branching rules

The derivation of branching rules given in this paper rests heavily on the results given by KLW and BKW making extensive use of the properties of S functions, S-function series and the properties of difference characters. A proof for the branching rules for $SO_{2k}\downarrow U_k$ has been sketched in BKW. Derivations for subgroups of O_n have been given by King (1975, 1982). The results given here are found in a similar manner. The decompositions of the vector [1], basic spinors Δ, $\Delta_\pm$ and Δ'', and of $\Box$, $\Box_\pm$ and $\Box''$ are first determined making use of identities given in KLW and BKW and the properties of weight spaces. S-function series are then symbolically manipulated as in BKW to finally yield the desired results.

In presenting our results, attention has been given to producing algorithms that avoid overcounting problems. These problems are signalled by the occurrence of explicit phase factors in the formulae. In the case of the spin irreps $[\Delta;\lambda]_\pm$ it is always possible to produce a final result free from explicit overcounting or requiring any use of difference characters. A similar situation holds for tensor irreps $[\lambda]$ having fewer than k parts.

For tensor irreps $[\lambda]_\pm$ having k parts (or equivalently $[\square; \lambda]_\pm$) it is possible to produce formulae that avoid any use of difference characters but only at the expense of introducing severe overcounting. In these cases difference character methods become much more efficient and avoid explicit overcounting. The branching rules for the k-part reducible representation $[\lambda]$ and for the difference character $[\lambda]''$ are obtained and then the results combined noting that

$$[\lambda] = [\lambda]_+ + [\lambda]_-, \qquad [\lambda]'' = [\lambda]_+ - [\lambda]_- \tag{8}$$

and

$$[\lambda]_\pm = \tfrac{1}{2}([\lambda] \pm [\lambda]''). \tag{9}$$

The formulae given herein follow the notation given in BKW and the reader is referred to that paper for precise examples of the various letter designated S-function series (especially equation (4.5) and table 6 of BKW). These results are all incorporated in a computer package SCHUR devised by one of us (Black 1982). This program has been used to produce the examples given in this paper.

5. Branching rule for $SO_{2k} \downarrow SU_k \times U_1$

The branching rule for $SO_{2k} \downarrow U_k$ was given in BKW. The corresponding rule for $SO_{2k} \downarrow SU_k \times U_1$ is given in table 4(a). Throughout we use $\omega_\alpha, \omega_\beta, \ldots$ to stand for the sum of parts of the corresponding partitions $\alpha, \beta, \ldots$.

Consider the 560-dimensional irrep of $SO_{10}[\Delta; 1^2]_+$ under $SO_{10} \downarrow SU_5 \times U_1$. Referring to table 4($a$) we have

$$\begin{aligned}[\Delta; 1^2]_+ \downarrow & \sum_{s,\zeta,\beta} \{\overline{\zeta \cdot 1^{2s}}; 1^2/\zeta\beta\} \times \{2 - 2\omega_\zeta - 2s - \omega_\beta + \tfrac{5}{2}\} \\ &= \sum_{s,\zeta} (\{\overline{\zeta \cdot 1^{2s}}; 1^2/\zeta\} \times \{\tfrac{9}{2} - 2\omega_\zeta - 2s\} + \{\overline{1^{2s}}\} \times \{\tfrac{3}{2} - 2s\})\end{aligned} \tag{10a}$$

where β is restricted to $\{0\}$ and $\{1^2\}$ and (6) has been used. Summing over ζ gives (10a) as

$$\sum_s (\{\overline{1^{2s}}; 1^2\} \times \{\tfrac{9}{2} - 2s\} + \{\overline{1 \cdot 1^{2s}}; 1\} \times \{\tfrac{5}{2} - 2s\} + \{\overline{1^2 \cdot 1^{2s}}\} \times \{\tfrac{1}{2} - 2s\}) \tag{10b}$$

with ζ being restricted to the partitions $\{0\}$, $\{1\}$ and $\{1^2\}$. The summation over s is restricted by the modification rules to give (10b) as

$$\begin{aligned}\{1^2\} \times \{\tfrac{9}{2}\} + \{\overline{1^2}; 1^2\} \times \{\tfrac{5}{2}\} + \{\overline{1^4}; 1^2\} \times \{\tfrac{1}{2}\} + \{\overline{1^6}; 1^2\} \times \{-\tfrac{3}{2}\} \\ + \{\bar{1} \cdot 1\} \times \{\tfrac{5}{2}\} + \{\overline{1 \cdot 1^2}; 1\} \times \{\tfrac{1}{2}\} + \{\overline{1 \cdot 1^4}; 1\} \times \{-\tfrac{3}{2}\} \\ + \{\overline{1 \cdot 1^6}; 1\} \times \{-\tfrac{7}{2}\} + \{\overline{1^2}\} \times \{\tfrac{1}{2}\} + \{\overline{1^2 \cdot 1^2}\} \times \{-\tfrac{3}{2}\} + \{\overline{1^2 \cdot 1^4}\} \times \{-\tfrac{7}{2}\}.\end{aligned} \tag{10c}$$

The SU_5 modification rules in table 3 are now used. Typically $\{\overline{1^2}; 1^2\} = \{2^2 1\}$,

$\{\overline{1^6}; 1^2\} = -\{\overline{1^4}\} = -\{1\}$, $\{\overline{1^4}; 1^2\} = 0$, $\{\overline{1 \cdot 1^2}; 1\} = \{\overline{1^3}; 1\} + \{\overline{21}; 1\} = \{21\} + \{32^21\}$, etc, to give finally

$$\begin{aligned}[\Delta; 1^2]_+ \downarrow \{1^2\} \times \{\tfrac{9}{2}\} &+ (\{2^21\} + \{21^3\} + \{0\}) \times \{\tfrac{5}{2}\} \\ &+ (\{32^21\} + \{21\} + 2\{1^3\}) \times \{\tfrac{1}{2}\} + (\{31^3\} + \{2^3\} \\ &+ \{2^21^2\} + \{1\}) \times \{-\tfrac{3}{2}\} + \{21^2\} \times \{-\tfrac{7}{2}\}.\end{aligned}$$

In a similar manner we find for the 3696-dimensional irrep $[\square; 1^2]_+$ of SO_{10}

$$\begin{aligned}[\square; 1^2]_+ \downarrow \{1^2\} \times \{7\} &+ (\{2^21\} + \{21^3\} + \{0\}) \times \{5\} \\ &+ (\{3^22\} + \{32^21\} + \{21\} + 2\{1^3\}) \times \{3\} + (\{43^21\} \\ &+ \{321\} + \{31^3\} + 2\{2^3\} + \{2^21^2\} + \{1\}) \times \{1\} \\ &+ (\{42^21\} + \{3^3\} + \{3^221\} + \{31\} + 2\{21^2\}) \times \{-1\} \\ &+ (\{41^3\} + \{32^2\} + \{321^2\} + \{2\}) \times \{-3\} + \{31^2\} \times \{-5\}.\end{aligned}$$

We note that in the case of $SO_{2k} \downarrow U_k$ and $SO_{2k} \downarrow SU_k \times U_1$ it has been possible to produce formulae even for the k-part tensor irrep that obviate the need to use difference characters.

Table 4. Branching rules.

(*a*) SO_{2k} $\downarrow$	$SU_k \times U_1$
Δ	$\sum_s \{1^{k-s}\} \times \{\frac{1}{2}k - s\}$
Δ''	$\sum_s (-1)^s \{1^{k-s}\} \times \{\frac{1}{2}k - s\}$
$\Delta_\pm$	$\sum_{s_\pm} \{1^{k-s_\pm}\} \times \{\frac{1}{2}k - s_\pm\}$
$\square$	$\sum_{s,t} \{2^{k-s-2t}1^{2t}\} \times \{k - 2s - 2t\}$
$\square''$	$\sum_{s,t} (-1)^s \{2^{k-s-2t}1^{2t}\} \times \{k - 2s - 2t\}$
[a] $\square_\pm$	$\sum_{s_\pm,t} \{2^{k-s_\pm-2t}1^{2t}\} \times \{k - 2s_\pm - 2t\}$
$[\Delta; \lambda]_+$	$\sum_{s,\zeta,\beta} \{\overline{\zeta \cdot 1^{2s}}; \lambda/\zeta\beta\} \times \{\omega_\lambda - 2\omega_\zeta - 2s - \omega_\beta + \frac{1}{2}k\}$
$[\Delta; \lambda]_-$	$\sum_{s,\zeta,\beta} \{\overline{\zeta \cdot 1^{2s+1}}; \lambda/\zeta\beta\} \times \{\omega_\lambda - 2\omega_\zeta - 2s - 1 - \omega_\beta + \frac{1}{2}k\}$
$[\lambda]$	$\sum_{\zeta,\beta} \{\bar{\zeta}; \lambda/\zeta\beta\} \times \{\omega_\lambda - 2\omega_\zeta - \omega_\beta\}$ $\quad p < k$
[b] $[\square; \lambda]_\pm$	$\frac{1}{2} \sum_{\zeta,\omega,\beta} (1 \pm (-1)^{k+q}) \{\bar{\zeta}; (\lambda/\zeta\beta) \cdot \tilde{\omega}\} \times \{\omega_\lambda - 2\omega_\zeta - \omega_\beta + \omega_\omega - k\}$

[a] s_+ represents positive *even* integers and s_- positive odd integers.
[b] q is the second part of the two-part partitions (ω) defined in KLW § 2.
The partitions β are any compatible member of the B series of S functions (BKW, 4.5*a*).

Table 4. (continued)

(*b*) SO_{2k}	$\downarrow$	SO_{2k-1}
Δ	$\downarrow$	2Δ
Δ''	$\downarrow$	0
$\Delta_\pm$	$\downarrow$	Δ
$\Box$	$\downarrow$	$2[1^{k-1}]$
$\Box''$	$\downarrow$	0
$\Box_\pm$	$\downarrow$	$[1^{k-1}]$
$[\Delta; \lambda]_\pm$	$\downarrow$	$[\Delta; \lambda/M]$
$[\lambda]$	$\downarrow$	$[\lambda/M]$[a]
$[\lambda]_\pm$	$\downarrow$	$\frac{1}{2}[\lambda/M]$[b]

[a] $\lambda_k = 0$, [b] $\lambda_k \neq 0$.

(*c*) SO_{2k} $\downarrow$	$SO_{2k-2} \times U_1$
Δ	$\Delta \times (\{\frac{1}{2}\} + \{-\frac{1}{2}\})$
Δ''	$\Delta'' \times (\{\frac{1}{2}\} - \{-\frac{1}{2}\})$
$\Delta_\pm$	$\Delta_\pm \times \{\frac{1}{2}\} + \Delta_\mp \times (-\frac{1}{2}\}$
$\Box$	$\Box \times (\{1\}) + \{-1\}) + 2[1^{k-2}] \times \{0\}$
$\Box''$	$\Box'' \times (\{1\} - \{-1\})$
$\Box_\pm$	$\Box_\pm \times \{1\} + \Box_\mp \times \{-1\} + [1^{k-2}] \times [0]$
$[\Delta; \lambda]_\pm$	$\sum_{s,t} ([\Delta; \lambda/s \cdot t]_\pm \times \{s-t+\frac{1}{2}\} + [\Delta; \lambda/s \cdot t]_\mp \times \{s-t-\frac{1}{2}\})$
$[\lambda]$	$\sum_{s,t} [\lambda/s \cdot t] \times \{s-t\}$
$[\lambda]''$	$\sum_{s,t} [\Box; \lambda/1^k \cdot s \cdot t]'' \times (\{s-t+1\} - \{s-t-1\})$

(*d*) $SO_{2(p+q)}$ $\downarrow$	$SO_{2p} \times SO_{2q}$	
Δ	$\Delta \times \Delta$	
Δ''	$\Delta'' \times \Delta''$	
$\Delta_\mp$	$\Delta_+ \times \Delta_\pm + \Delta_- \times \Delta_\mp$	
$\Box$	$\Box \times \Box + 2 \sum_{r=0}^{q-1} [1^{p-q+r}] \times [1^r]$	
$\Box''$	$\Box'' \times \Box''$	
$\Box_\pm$	$\Box_+ \times \Box_\pm + \Box_- \times \Box_\mp + \sum_{r=0}^{p-1} [1^{p-q+r}] \times [1^r]$	$p \geqslant q$
$[\Delta; \lambda]_\pm$	$\sum_{\rho,\zeta} ([\Delta; \lambda/\zeta]_+ \times [\Delta; \zeta/\rho]_{\pm(-)}\omega_\rho + [\Delta; \lambda/\zeta]_- \times [\Delta; \zeta/\rho]_{\mp(-\zeta)}\omega_\rho)$	
$[\lambda]$	$\sum_{\zeta} [\lambda/\zeta] \times [\zeta/D]$	
$[\lambda]''$	$\sum_{\zeta} [\Box; \lambda/1^{p+q}/\zeta]'' \times [\Box; \zeta/B]''$	

Table 4. (continued)

(e) $SO_{2(p+q+1)} \downarrow$	$SO_{2p+1} \times SO_{2q+1}$
Δ	$2(\Delta \times \Delta)$
Δ''	0
$\Delta_\pm$	$\Delta \times \Delta$
$\square$	$2([1^p] \times [1^q])$
$\square''$	0
$\square_\pm$	$[1^p] \times [1^q]$
$[\Delta; \lambda]_\pm$	$\sum_{\zeta,\rho} [\Delta; \lambda/\zeta] \times [\Delta; \zeta/\rho]$
$[\lambda]$	$\sum_{\zeta} [\lambda/\zeta] \times [\zeta/D]^{a}$
$[\lambda]_\pm$	$\frac{1}{2}\sum_{\zeta} [\lambda/\zeta] \times [\zeta/D]^{b}$

[a] $\lambda_{p+q+1} = 0$.
[b] $\lambda_{p+q+1} \neq 0$.

6. Branching rules for $SO_{2k} \downarrow SO_{2k-1}$ and $SO_{2k} \downarrow SO_{2k-2} \times U_1$

These branching rules are given in tables 4(*b*) and 4(*c*). In the case of $SO_{10} \downarrow SO_9$ we have

$$[\Delta; 21]_\pm \downarrow [\Delta; 21] + [\Delta; 2] + [\Delta; 1^2] + [\Delta; 1]$$

and

$$[\Delta; 1^5]_\pm \downarrow [\Delta; 1^5] + [\Delta; 1^4]$$

but for SO_9 the modification rules give $[\Delta; 1^5] = 0$ and hence

$$[\Delta; 1^5]_\pm \downarrow [\Delta; 1^4].$$

In a similar way under $SO_{10} \downarrow SO_9$

$$[21^4]_\pm \downarrow \tfrac{1}{2}([21^4] + [21^3] + [1^5] + [1^4]) = [21^3] + [1^4].$$

The $SO_{2k} \downarrow SO_{2k-2} \times U_1$ branching rule for the spinor irrep of SO_{2k} may be evaluated without recourse to difference characters. Thus

$$\begin{aligned}
[\Delta; 21^4]_+ \downarrow & \sum_{s,t} ([\Delta; 21^4/st]_+ \times \{s - t + \tfrac{1}{2}\} + [\Delta; 21^4/st]_- \times \{s - t - \tfrac{1}{2}\}) \\
& = \sum_t ([\Delta; 21^4/t]_+ \times \{\tfrac{1}{2} - t\} + ([\Delta; 21^3/t]_+ + [\Delta; 1^5]_+) \\
& \quad \times \{\tfrac{3}{2} - t\} + [\Delta; 1^4/t]_+ \times \{\tfrac{5}{2} - t\} + \ldots \\
& = [\Delta; 21^4]_+ \times \{\tfrac{1}{2}\} + ([\Delta; 21^3]_+ + [\Delta; 1^5]_+) \times \{-\tfrac{1}{2}\} + [\Delta; 1^4]_+ \times \{-\tfrac{3}{2}\} \\
& \quad + ([\Delta; 21^3]_+ + [\Delta; 1^5]_+) \times \{\tfrac{3}{2}\} + ([\Delta; 21^2]_+ + [\Delta; 1^4]_+) \times \{\tfrac{1}{2}\} + [\Delta; 1^3]_+ \times \{-\tfrac{1}{2}\} \\
& \quad + [\Delta; 1^4]_+ \times \{\tfrac{5}{2}\} + [\Delta; 1^3]_+ \times \{\tfrac{3}{2}\} + \ldots
\end{aligned}$$

$$= [\Delta;\, 1^4]_+ \times \{\tfrac{5}{2}\} + ([\Delta;\, 21^3]_+ + [\Delta;\, 1^5]_+ + [\Delta;\, 1^3]_+) \times \{\tfrac{3}{2}\} + ([\Delta;\, 21^4]_+ + [\Delta;\, 21^2]_+ + 2[\Delta;\, 1^4]_+) \times \{\tfrac{1}{2}\} + ([\Delta;\, 21^3]_+ + [\Delta;\, 1^5]_+ + [\Delta;\, 1^3]_+) \times \{-\tfrac{1}{2}\} + [\Delta;\, 1^4]_+) \times \{-\tfrac{3}{2}\} + \ldots$$

where the extra terms ... are obtained by replacing the + subscript by a − subscript in the printed terms and subtracting 1 from each U_1 irrep label. The above result holds for all $k \geqslant 6$ without modification. For $SO_{10}\downarrow SO_8 \times U_1$ the SO_8 irrep having five parts must be modified to give

$$[\Delta;\, 1^5]_+ = -[\Delta;\, 1^4]_-, \qquad [\Delta;\, 1^5]_- = -[\Delta;\, 1^4]_+,$$
$$[\Delta;\, 21^4]_+ = -[\Delta;\, 21^3]_-, \qquad [\Delta;\, 21^4]_- = -[\Delta;\, 21^3]_+$$

to yield for $SO_{10}\downarrow SO_8 \times U_1$

$$[\Delta;\, 21^4]_+ \downarrow [\Delta;\, 1^4]_+ \times \{\tfrac{5}{2}\} + ([\Delta;\, 21^3]_+ + [\Delta;\, 1^3]_+) \times \{\tfrac{3}{2}\} + ([\Delta;\, 21^2]_+ + [\Delta;\, 1^4]_+ + [\Delta;\, 1^3]_-) \times \{\tfrac{1}{2}\} + ([\Delta;\, 1^3]_+ + [\Delta;\, 21^2]_- + [\Delta;\, 1^4]_-) \times \{-\tfrac{1}{2}\} + ([\Delta;\, 21^3]_- + [\Delta;\, 1^3]_-) \times \{-\tfrac{3}{2}\} + [\Delta;\, 1^4]_- \times \{-\tfrac{5}{2}\}.$$

The branching rules for the k-part tensor irreps of $SO_{2k}\downarrow SO_{2k-2} \times U_1$ are found by first restricting $[\lambda]$ and $[\lambda]''$ and then combining the results using (9). Thus for $SO_{10}\downarrow SO_8 \times U_1$:

$$[21^4]\downarrow [1^4] \times \{2\} + [21^3] \times \{1\} + 2[1^4] \times \{0\} + [21^3] \times \{-1\} + [1^4] \times \{-2\}$$

and

$$[21^4]''\downarrow [1^4]'' \times \{2\} + [21^3]'' \times \{1\} - [21^3]'' \times \{-1\} - [1^4]'' \times \{-2\}$$

and hence

$$[21^4]_+ = \tfrac{1}{2}([21^4] + [21^4]'')\downarrow [1^4]_+ \times \{2\} + [21^3]_+ \times \{1\} + ([1^4]_+ + [1^4]_-) \times \{0\} + [21^3]_- \times \{-1\} + [1^4]_- \times \{-2\}.$$

7. Branching rules for $SO_{2(p+q)}\downarrow SO_{2p} \times SO_{2q}$

These branching rules are given in table 4(*d*). Their evaluation follows closely the methods used in the preceding section. Typically, for $SO_{10}\downarrow SO_6 \times SO_4$ we have

$$[\Delta;\, 2]_+ \downarrow [\Delta;\, 0]_- \times [\Delta;\, 2]_- + [\Delta;\, 0]_+ \times [\Delta;\, 2]_+ + [\Delta;\, 1]_- \times [\Delta;\, 1]_- + [\Delta;\, 1]_+ \times [\Delta;\, 1]_+ + [\Delta;\, 0]_- \times [\Delta;\, 1]_+ + [\Delta;\, 0]_+ \times [\Delta;\, 1]_- + [\Delta;\, 2]_- \times [\Delta;\, 0]_- + [\Delta;\, 2]_+ \times [\Delta;\, 0]_+ + [\Delta;\, 1]_- \times [\Delta;\, 0]_+ + [\Delta;\, 1]_+ \times [\Delta;\, 0]_- + [\Delta;\, 0]_+ \times [\Delta;\, 0]_- + [\Delta;\, 0]_+ [\Delta;\, 0]_+$$

and

$$[21^4]_+ \downarrow [21^2]_+ \times [1^2]_+ + [21^2]_- \times [1^2]_- + [21] \times [1] + [2] \times [0] + [1^3]_+ \times ([21]_+ + [1]) + [1^3]_- \times ([21]_- + [1]) + [1^2] \times ([2] + [1^2]_+ + [1^2]_- + [0]) + [1] \times [1].$$

8. Branching rules for $SO_{2(p+q+1)} \downarrow SO_{2p+1} \times SO_{2q+1}$

These branching rules are summarised in table 4(*e*) and present no special problems in their applications. Typically for $SO_{12} \downarrow SO_7 \times SO_5$

$$[\Delta; 21]_+ \downarrow [\Delta; 0] \times ([\Delta; 21] + [\Delta; 2] + (\Delta; 1^2] + 2[\Delta; 1] + [\Delta; 0])$$
$$+ [\Delta; 1] \times ([\Delta; 2] + [\Delta; 1^2] + 2[\Delta; 1] + 2[\Delta; 0])$$
$$+ [\Delta; 2] \times ([\Delta; 1] + [\Delta; 0]) + [\Delta; 1^2] \times ([\Delta; 1] + [\Delta; 0]) + [\Delta; 21] \times [\Delta; 0]$$

and

$$[\Box; 1]_+ \downarrow [1^3] \times ([21] + [1^2] + [1]) + [21^2] \times [1^2] + [21] \times [1]$$
$$+ [1^2] \times ([2] + [1^2] + [0]) + [1] \times [1] + [2] \times [0].$$

9. The inverse of restriction $\uparrow_r$

Branching rules are normally associated with the group–subgroup restriction $G \downarrow H$. The inverse of restriction $G \uparrow_r H$ has been considered by King (1975) and finds applications in supersymmetry theories (cf Curtright 1982a, b). Clearly the operation $\uparrow_r$ is defined only over classes common to G and H and is not necessarily unique. Important cases of $\uparrow_r$ for rotation groups are given in table 5. The inverse of restriction

Table 5. Inverses of restriction for rotation groups.

SO_{2k}	$\uparrow_r$	SO_{2k+1}
$[\Delta; \lambda]$		$[\Delta; \lambda/L]$
$[\lambda]$		$[\lambda/L]$
SO_{2k+1}	$\uparrow_r$	SO_{2k+2}
$[\Delta; \lambda]$		$\sum_m [\Delta; \lambda/1^m R]_{\pm(-)^m}$
$[\lambda]$		$[\lambda/L]$
SO_{2k}	$\uparrow_r$	SO_{2k+2}
$[\Delta; \lambda]$		$\sum_m [\Delta; \lambda/1^m R]_{\pm(-)^m}$
$[\lambda]$		$[\lambda/LL]$
SO_{2k}	$\uparrow_r$	$SO_{2k+2} \times SU_2$
$[\lambda]$		$\sum_\zeta (-1)^\zeta [\lambda/\zeta] \times \{\tilde{\zeta}\}$
$[\Delta; \lambda]$		$\sum_{m,\zeta} (-1)^\zeta [\Delta; \lambda/1^m \zeta P]_{\pm(-)^m} \times \{\tilde{\zeta}\}$
SO_{2k+1}	$\uparrow_r$	$SO_{2k+3} \times SU_2$
$[\lambda]$		$\sum_\zeta (-1)^\zeta [\Delta/\zeta] \times \{\tilde{\xi}\}$
$2[\Delta; \lambda]$		$\sum_\zeta (-1)^\zeta [\Delta; \lambda/\zeta] \times \{\tilde{\zeta}\}$

for SO_{2k} requires the use of reducible representations in the case of spinor representations or tensor representations having k parts. Thus for $SO_{2k}\uparrow_r SO_{2k+1}$ we have

$$[\Delta;\lambda]=[\Delta;\lambda]_+ + [\Delta;\lambda]_- \uparrow_r [\Delta;\lambda/L].$$

For example

$$[\Delta;21]\uparrow_r[\Delta;21]-[\Delta;2]-[\Delta;1^2]+[\Delta;1].$$

We note that under $\uparrow_r$ the need for modification rules never arises.

Under $SO_{2k+1}\uparrow_r SO_{2k+2}$ we may take either $\Delta\uparrow_r\Delta_+$ or $\Delta\uparrow_r\Delta_-$. The choice of $\pm$ in table 5 reflects this ambiguity.

The operations $SO_{2k}\uparrow_r SO_{2k+2}\times SU_2$ and $SO_{2k+1}\uparrow_r SO_{2k+3}\times SU_2$ are somewhat novel. The labels of the irreps of SU_2 can all be reduced to one part by noting the equivalence

$$\{\lambda_1\lambda_2\}=\{\lambda_1-\lambda_2\}.$$

A given SU_2 irrep $\{a\}$ is of dimension $a+1$. Thus the 'group' SU_2 plays the role of a 'multiplicity counting group' (Wybourne 1983).

The S-function series P, L and R are defined in KLW and BKW. Note that $R=PL$ and

$$LL=\sum_{n=0}\sum_{m=0}^{[n/2]}(-1)^n(n+1-2m)\{2^m1^{n-2m}\}$$
$$=\{0\}-2\{1\}+3\{1^2\}+\{2\}-4\{1^3\}-2\{21\}+5\{1^4\}+3\{21^2\}+\{2^2\}-\dots.$$

For $SO_{10}\uparrow SO_{12}$ we have

$$[21^4]_+ + [21^4]_- \uparrow_r [21^4/LL]$$
$$=[21^4]+3[21^2]+4[1^4]+5[2]+8[1^2]+5[0]$$
$$-(2[21^3]+2[1^5]+4[21]+6[1^3]+10[1])$$

which is identical to Curtright's (1982a, b) result. Curtright divides his SO_{12} irrep into primary gauge and accompanying ghost fields. It is interesting to note that for boson states his primary gauge fields correspond to the terms in $[\lambda/K]$ where K is the S-function series

$$K=\{0\}+\{2\}+\{2^2\}+\{2^3\}+\dots. \tag{11}$$

The fermionic spinor cases are complicated by the presence of the $\pm$ subscripts that distinguish conjugate irreps. However, the distinction is effectively lost under $\uparrow_r$. As a consequence a reducible representation $[\Delta;\lambda]$ of SO_{2k} may be expanded to SO_{2k+2} by simply writing

$$[\Delta;\lambda]\uparrow_r[\Delta;\lambda/LL]_\pm \tag{12}$$

which is significantly simpler than the result given in table 5. It is important to realise that while the various results for $SO_{2k}\uparrow_r SO_{2k+2}$ appear to involve different representations of SO_{2k+2} in fact they still preserve the various dimension and Dynkin index sum rules. Thus for example, table 5 leads to the expansion of $[\Delta;21]$ under

$SO_{10}\uparrow_r SO_{12}$ as

$$[\Delta; 21]\uparrow_r [\Delta; 21]_+ + [\Delta; 2]_- + [\Delta; 1^2]_- + 10[\Delta; 1]_+ + 9[\Delta; 0]_- - (3[\Delta; 2]_+ + 3[\Delta; 1]_+ + 6[\Delta; 1]_- + 11[\Delta; 0]_+)$$

while use of (12) gives

$$[\Delta; 21]\uparrow_r [\Delta; 21]_+ + 4[\Delta; 1]_+ - 2([\Delta; 2]_+ + [\Delta; 1^2]_+ + [\Delta; 0]_+).$$

Curtwright (1982a, b) gives

$$[\Delta; 21]\uparrow_r [\Delta; 21]_- + [\Delta; 2]_+ + [\Delta; 1^2]_+ + 2[\Delta; 1]_- + [\Delta, 0]_+ - (2[\Delta; 2]_- + 2[\Delta; 1^2]_- + 2[\Delta; 1]_+ + 2[\Delta; 0]_- + [\Delta; 2]_+ + [\Delta; 1^2]_+ + [\Delta; 0]_+) + 2[\Delta; 1]_- + 2[\Delta; 1]_+.$$

Each of the above results satisfies the dimension and Dynkin index sum rules. Since these sum rules do not distinguish members of a conjugate pair they can be satisfied by any arrangement of the $\pm$ subscripts.

Curtright's results were checked using the program SCHUR to perform the restriction $SO_{12}\downarrow SO_2\times SO_{10}$ which is a special case of the results given in table 4(*e*) for $SO_{2(p+q)}\downarrow SO_{2p}\times SO_{2q}$. Since SCHUR can handle strings of irreps this proved a simple task and confirmed all of Curtright's results.

10. Symmetrised powers of basic spinor irreps

The resolution of the symmetrised powers of the basic spin irreps Δ and $\Delta_\pm$ of SO_{2k+1} and SO_{2k} respectively has been discussed by KLW who gave a complete algorithm for up to the fourth power for all k (see also King and Wybourne 1982). In general these resolutions correspond to evaluation of the spinor plethysms (cf Littlewood 1947, 1948, 1950)

$$\Delta\otimes\{\lambda\} \qquad \text{for } SO_{2k+1} \tag{13a}$$

and

$$\Delta_\pm\otimes\{\lambda\} \qquad \text{for } SO_{2k} \tag{13b}$$

where for the pth power in Δ (or $\Delta_\pm$) $\{\lambda\}$ is a partition of p.

Equivalently, the evaluation of the above spinor plethysms can be regarded as an evaluation of the branching rules for the unitary group irrep $\{\lambda\}$ under the restriction

$$U_{2^k}\downarrow SO_{2k+1} \qquad \text{where} \qquad \{1\}\downarrow\Delta \tag{14a}$$

or

$$U_{2^{k-1}}\downarrow SO_{2k} \qquad \text{where} \qquad \{1\}\downarrow\Delta_\pm \tag{14b}$$

where in (14*b*) if we choose $\{1\}\downarrow\Delta_+$ then $\{\bar{1}\}\downarrow\Delta_-$. The plethysms $\Delta\otimes\{1^n\}$ or $\Delta_\pm\otimes\{1^n\}$ correspond to the reduction of an antisymmetric tensor $\{1^n\}$ under the rotation group—a problem of considerable relevance in supersymmetric theories (Bergshoeff and DeRoo 1982). Such a problem may also be viewed as taking a set of fermions spanning Δ (or $\Delta_\pm$) and forming all possible antisymmetric states. The terms with n *even* yield tensor irreps while those with n *odd* yield spinor irreps. The complete set of states span the vector irrep $\{1\}$ of $SU_{2\alpha}$ where $\alpha = 2^k$ for SO_{2k+1} or $2^{(k-1)}$ for SO_{2k}.

Under the restriction $SU_{2\alpha} \downarrow SO_{\alpha+1}$ we have (Wybourne 1974)

$$\{1\} \downarrow \Delta. \tag{15}$$

Further restriction to SO_α gives

$$\{1\} \downarrow \Delta \downarrow \Delta_+ + \Delta_-. \tag{16}$$

These two irreps, Δ_+ and Δ_-, have the same dimensions and set of Dynkin indices. Still further restriction to $SU_{2^{k-1}}$ (or $SU_{2^{k-2}}$) is possible using the $SO_{2p} \downarrow U_p$ rule to give

$$\Delta_+ \downarrow \{0\} + \{1^2\} + \ldots + \{1^P\} \tag{17a}$$

and

$$\Delta_- \downarrow \{1\} + \{1^3\} + \ldots + \{1^{P-1}\} \tag{17b}$$

giving equality between the boson and fermion states with corresponding assurance of equal Dynkin index summations in both sectors (Curtright 1982a, Ferrara *et al* 1981).

Finally the reduction $SU_{2^{k-1}} \downarrow SO_{2k+1}$ (or $SU_{2^{k-2}} \downarrow SO_{2k}$) may be made by evaluating the spinor plethysm $\Delta \otimes \{1^n\}$ (or $\Delta_\pm \otimes \{1^n\}$).

In the specific case of the 16-dimensional irrep Δ_+ of SO_{10} we have the group chain

$$U_{65536} \downarrow SO_{33} \downarrow SO_{32} \downarrow SU_{16} \downarrow SO_{10}.$$

Under $SO_{32} \downarrow SU_{16}$ we have

$$\Delta_+ \downarrow \{0\} + \{1^2\} + \{1^4\} + \ldots + \{1^{16}\} \tag{18a}$$

and

$$\Delta_- \downarrow \{1\} + \{1^3\} + \{1^5\} + \ldots + \{1^{15}\}. \tag{18b}$$

The reductions for $SU_{16} \downarrow SO_{10}$ may be readily deduced by first using KLW to evaluate $\Delta_+ \otimes \{1^n\}$ for $n \leq 4$. The case for $n = 5$ to 8 can be constructed from these results using the results of BKW to evaluate Kronecker products. We give the results for $n = 0$ to 8 in table 6. The terms for $n > 8$ can be found by recalling the involutory outer-automorphism † for SO_{2k} which gives

$$\Delta_+ \otimes \{1^{16-n}\} = (\Delta_+ \otimes \{1^n\})^\dagger \tag{19}$$

recalling that under † we have

$$([\lambda]_\pm)^\dagger = [\lambda]_\mp. \tag{20}$$

The results in table 6 are in agreement with the results of Bergshoeff and DeRoo (1982) but with the important difference that the ambiguity between the $[\lambda]_\pm$ irreps is removed.

We note that once $\Delta_+ \otimes \{1^n\}$ over all compatible n is known then any other symmetrised power of Δ_+, say $\Delta_+ \otimes \{\lambda\}$, may be evaluated since any S function $\{\lambda\}$ may be expanded as a sum of products of the S functions $\{1^x\}$ and furthermore

$$\Delta_+ \otimes ABC \ldots = (\Delta_+ \otimes A)(\Delta_+ \otimes B)(\Delta_+ \otimes C) \ldots. \tag{21}$$

We note that Curtright's (1982) trial and error calculation of the $D = 12$ local fields and their massless SO_{10} on-shell states (his table I) is essentially just the content of our table 6.

Table 6. Branching rules for the antisymmetric tensor irreps of $SU_{16}\downarrow SO_{10}$.

$D_{\{\lambda\}}$	SU_{16} $\downarrow$	SO_{10}
16	$\{1\}$	$[\Delta; 0]_+$
120	$\{1^2\}$	$[1^3]$
560	$\{1^3\}$	$[\Delta; 1^2]_-$
1 820	$\{1^4\}$	$[21^4]_- + [2^2]$
4 368	$\{1^5\}$	$[\Delta; 21]_- + [\Delta; 1^5]_-$
8 624	$\{1^6\}$	$[2^21^3]_- + [31^2]$
11 440	$\{1^7\}$	$[\Delta; 21^2]_- + [\Delta; 3]_+$
12 870	$\{1^8\}$	$[4] + [2^3] + [31^3]$

11. Symmetrised powers of SO_{11}

If the symmetrised powers $\Delta_\pm \otimes \{1^n\}$ are known for SO_{10} then we can obtain the corresponding resolutions for SO_{11} $\Delta \otimes \{1^n\}$. These may be evaluated by noting that under $SO_{11}\downarrow SO_{10}$ we have $\Delta\downarrow\Delta_+ + \Delta_-$. The terms in $(\Delta_+ + \Delta_-)\otimes\{1^n\}$ for SO_{10} may be evaluated by noting that

$$(\Delta_+ + \Delta_-)\otimes\{1^n\} = \sum_{x=0}^{n} (\Delta_+ \otimes \{1^x\})(\Delta_- \otimes \{1^{n-x}\})$$

$$= \sum_{x=0}^{n} (\Delta_+ \otimes \{1^x\})(\Delta_+ \otimes \{1^{n-x}\})^\dagger. \quad (22)$$

The relevant Kronecker products in SO_{10} may be evaluated and collected together, replacing each $[\lambda]_+ + [\lambda]_-$ pair by just $[\lambda]$, and placing the irrep in order of descending weight. Then under $SO_{11}\downarrow SO_{10}$ we have

$$[\lambda]\downarrow[\lambda]+\ldots$$

where $[\lambda]$ is the leading term in the reduction. The SO_{10} content of the irrep of SO_{11} covering the highest weight SO_{10} irrep is removed and then the highest weight of the remaining SO_{10} irreps determined and the process continued. In this way it was possible to resolve $\Delta\otimes\{1^{16}\}$ for SO_{11} which is of dimension 601 080 390. This calculation was facilitated by the use of SCHUR which evaluates sums of Kronecker products and branching rules for strings of representations. Thus, in principle, we can state that it is now posssible to evaluate unambiguously arbitrary symmetrised powers for all irreps of SO_n for all $n \leqslant 11$.

Curtright (1982a, b) has made the remarkable observation that the terms arising in the antisymmetric powers of the spinor irrep of SO_9 are just the terms contained in the squaring of the SO_9 reducible representation $[1^3]+[2]+[\Delta; 1]$ implying that for SO_9

$$\sum_{n=0}^{16} \Delta\otimes\{1^n\} = ([1^3]+[2]+[\Delta; 1])^2.$$

Inspection of the terms in $\Delta\otimes\{1^{16}\}$ for SO_{11} show that his result cannot be extended to SO_{11} and would appear to be special to $n = 7$, 8 and 9.

The basic spinor of SO_{11} is symplectic and hence can be embedded in the Sp_{32} subgroup of SU_{32}. The graded Lie algebra OSp(32/1) has been used by D'Auria and

Fré (1982) in $D=11$ supergravity. Their analysis requires the resolution of the symmetric powers of the basic spinor of SO(1, 10) which is equivalent to determining $\Delta\otimes\{n\}$ in SO_{11}. We note that their result for $\Delta\otimes\{4\}$ (their equation 3.3) requires amending to

$$\Delta\otimes 4=[0]+[1^3]+[1^4]+[1^5]+[2]+[21]+[21^4]+[2^2]+[2^21^3]+[2^5].$$

Their erroneous result was compounded by their miscalculation of the dimension of the irrep $[2^5]$ which should have been 28 314. Their stated result 32 604 is coincidently that of the dimension of $[2^5]+[21^4]$.

D'Auria *et al* (1982a) have discussed the geometric structure of super Yang–Mills theory making extensive use of p-forms and the group SO(1, 9). It is worth noting that the dimensions of their two-form $\psi^\alpha\wedge\psi^\beta$ and their three-form $\psi^\alpha\wedge\psi^\beta\wedge\psi^\gamma$ are just those of the SO_{10} spinor plethysms $\Delta_+\otimes\{2\}$ and $\Delta_+\otimes\{3\}$. Their analysis can be made precise and unambiguous by analysis of the corresponding plethysms. In a similar manner the three-forms used by D'Auria *et al* (1982b) in their analysis of Bianchi identities may be given in terms of appropriate spinor plethysms. Thus equation (2.27) is equivalent to the resolution of $\Delta\otimes\{3\}$ for O_4. The dimensions of their three-forms in $D=4$ N-extended superspace are just those of the {3} irrep of U_{4N}. We note that for $N=8$ their result for the dimension of the three-form should be 5984 and not 6160.

12. Concluding remarks

The preceding remarks emphasise the need for unambiguous methods for evaluating branching rules and symmetrised powers of irreps. Patching dimensions can be fraught with errors (cf Castellani *et al* 1982). The methods oulined in this paper are unambiguous in their implementation and all exist as part of the computer package SCHUR.

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